Chapter 3.3 - Menger's theorem

Theorem (Menger 1927)
Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint A-B paths in G.

Let k be minimum number of vertices separating A from B in G. Proofs of Menger's theorem are trying to find k disjoint A-B paths in G. The other direction is obvious.

First proof by induction on the number of edges.
Idea: find a separator X between A and B such that $|X| = k$. Use induction to get paths A-X and $X - B$ that can be combined.

1: Show that Menger's theorem holds if G has no edges. (induction base)

Solution: This is trivial since now $|A \cap B| = k$, this is a test that we allow trivial paths in the overlap.

Let $e = xy$ be any edge in G.

2: Show that G/e contains a separator Y between A and B with $|Y| < k$.
Hint: Use induction.

Solution: If G/e contains k disjoint A-B paths, they are there also after decontraction with possible one tweaked. So G/e contains at most $k - 1$ A-B paths. By induction we conclude there is a cut Y with $|Y| < k$. This means that Y contains the contracted vertex as well.

Now $X := (Y \cap V(G)) \cup \{x, y\}$ is also A-B separator. Hence $|X| = k$.

3: Use induction on $G - e$ and finish the proof according to the outlined plan.

Solution: By induction there are k disjoint A-X paths in $G - e$, and similarly there are k disjoint X-B paths in $G - e$. As X separates A from B, these two path systems do not meet outside X, and can thus be combined to k disjoint A-B paths.

Let $\mathcal{P}$ and $\mathcal{Q}$ be a sets of disjoint A-B paths. We say that $\mathcal{Q}$ **exceeds** $\mathcal{P}$ if the set of vertices in A that lie on a path in $\mathcal{P}$ is a proper subset of the set of vertices in A that lie on a path in $\mathcal{Q}$, and likewise for B. Then, in particular, $|Q| \geqslant |P| + 1$.

Second proof: Goal is to show
If $\mathcal{P}$ is any set of fewer than k disjoint A-B paths in G, then G contains a set of $|\mathcal{P}| + 1$ disjoint A-B paths exceeding $\mathcal{P}$.

We proceed by induction on $|\mathcal{P}|$ and $|\bigcup \mathcal{P}|$. Assume $|\mathcal{P}| < k$.

4: Show that there exists A-B path R such that R does not contain any of the endpoints of $\mathcal{P}$ in B.

Solution: If we remove less than k endpoints of B, it does not separate A and B so there is still an A-B. Here figure is needed.

If R disjoint with all paths in $\mathcal{P}$, we can simply add R to $\mathcal{P}$ and we are done.

Let $P \in \mathcal{P}$ be the last path intersecting R, last intersection is x. Then xR has no intersection in $\mathcal{P}$ except x and ends in B.

Now the trick is to make B larger and find paths to the new B by induction.

Let $B' := B \cup xR \cup xP$. Let $\mathcal{P}' = (\mathcal{P} - P) \cup Px$.

5: Show that we can apply induction on A, B' and $\mathcal{P}'$.

Solution: Notice we made one path shorter and that is all.

By induction, there exists a A-B' path Q that is disjoint with $\mathcal{P}'$.

6: Finish the proof by inspecting endpoints on Q.

Solution: If Q ends in vertex q in xP, then we have paths $Q \cup qP$ and $Px \cup xR$. Otherwise $Q \cup qR$ or just Q and P work.